Connections on vector bundles and principal bundles

A *connection* on a vector bundle $E \to M$ is a linear map

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

satisfying a Leibniz rule:

$$\nabla_X(fs) = X(f)s + f\nabla_X s$$

for all $f \in C^\infty(M)$ and vector fields $X \in \Gamma(TM)$. (Here the notation $X(f)$ means $df(X)$).

What do connections look like in the simplest case: a trivial bundle $E = M \times V \to M$?
By choosing a trivialization $(E_1, \cdots, E_n)$ for $M \times V$ (e.g. choose a basis for V and let $E_1(x) = x \times e_1 \in x \times V \subset M \times V$), we can write a section $s : M \to M \times V$ as $s(x) = (s_1(x), \cdots, s_n(x))$ where $s_i \in C^\infty(M)$. A section $\sigma : M \to T^*M \otimes (M \times V)$ also splits as $\sigma = (\sigma_1(x), \cdots, \sigma_n(x))$, but here the σ_i are 1-forms on M. Let ∇ be a connection on $M \times V \to M$, and $s = (s_1, \cdots, s_n)$ a section. Then $s = s_1E_1 + \cdots + s_nE_n$ so by linearity and the Leibniz rule

$$\nabla_X s = \sum_{i=1}^{n} X(s_i)E_i + s_i\nabla_X E_i = ds(X) + As$$

where A is the matrix of 1-forms whose ith column is given by $\nabla_X E_i$. Therefore we can write $\nabla = d + A$. Conversely one can check that for any $n \times n$ matrix, A, whose entries are 1-forms, $d + A$ satisfies the requirements to be a connection on $M \times V^n$.
For a connection ∇ on a general vector bundle, described by $(\{U_\alpha\}, \{g_{\beta\alpha}\})$, the restriction $\nabla|_{U_\alpha}$ is a connection on the trivial bundle $U_\alpha \times V$ so it can be written as $d + A_\alpha$. However, generally the connection cannot be written globally in this way. If we consider the restriction of a section $s : M \to E$ to $U_\beta \cap U_\alpha$, the transition functions $g_{\beta\alpha}$ indicate how the connections on the trivial bundles $U_\alpha \times V$ and $U_\beta \times V$ relate:

$$g_{\beta\alpha}(ds_\alpha + A_\alpha s_\alpha) = (\nabla s)_\beta = ds_\beta + A_\beta s_\beta = d(g_{\beta\alpha}s_\alpha) + A_\beta g_{\beta\alpha}s_\alpha$$

$$ds_\alpha + A_\alpha s_\alpha = g_{\beta\alpha}^{-1}(g_{\beta\alpha}ds_\alpha + (dg_{\beta\alpha})s_\alpha + A_\beta g_{\beta\alpha}s_\alpha) = ds_\alpha + (g_{\beta\alpha}^{-1}dg_{\beta\alpha} + g_{\beta\alpha}^{-1}A_\beta g_{\beta\alpha})s_\alpha$$

So the relation is:

$$A_\alpha = g_{\beta\alpha}^{-1}dg_{\beta\alpha} + g_{\beta\alpha}^{-1}A_\beta g_{\beta\alpha}(*)$$

Conversely, every family $\{A_\alpha\}$ satisfying this condition glues together via a partition of unity subordinate to $\{U_\alpha\}$ to form a global connection on the vector bundle.

We can use this local method of defining connections to extend this to a definition of connections on principal bundles. For this condition on the A_α to make sense on a principal bundle, we first need to define some natural objects associated to the Lie group G.
For a Lie group G, let $\mathfrak{g}$ be its Lie algebra which is canonically identified with the tangent space to G at 1_G.
There is a naturally defined *adjoint action* of G given by $Ad_g : \mathfrak{g} \to \mathfrak{g}$ is the derivative at 1_G of $\Psi_g : G \to G$ defined by $\Psi_g(h) = ghg^{-1}$.
There is also a natural $\mathfrak{g}$-valued 1-form, ϕ called the Maurer-Cartan form, determined by
(1) $\phi(1_G)(X) = X \in T_{1_G}(X) = \mathfrak{g}$ and
(2) ϕ is left-invariant
For a collection $\{A_\alpha\}$ where $A_\alpha \in \Omega^1(U_\alpha) \otimes \mathfrak{g}$, we can form a connection on the principal G-bundle determined by $(\{U_\alpha\}, \{g_{\beta\alpha}\})$ if the relation (*) is satisfied:

$$A_\alpha = g_{\beta\alpha}^{-1}dg_{\beta\alpha} + g_{\beta\alpha}^{-1}A_\beta g_{\beta\alpha}(*)$$

where $g_{\beta\alpha}^{-1}dg_{\beta\alpha}$ is the pull-back under $g_{\beta\alpha}$ of the Maurer-Cartan form, and $g_{\beta\alpha}^{-1}A_\beta g_{\beta\alpha}$ is the adjoint action of $g_{\beta\alpha}$ acting on A_α.

Given a principal G-bundle, and a representation $\rho : G \to GL(V)$, there is an associated vector bundle to the principal G-bundle P: $P \times_\rho V$ defined by gluing cycles $(\{U_\alpha\}, \{\rho \circ g_{\beta\alpha}\})$. A connection on P defined by $\mathfrak{g}$-valued 1-forms, $\{A_\alpha\}$ induces a connection on $P \times_\rho V$ defined by $\{\rho_*(A_\alpha)\}$ where

$$\rho_* : T_1 G = \mathfrak{g} \to T_I(GL(V)) = End(V).$$

Curvature of a connection

Given a connection ∇ on a vector bundle $E \to M$, we can define its curvature F_∇ by

$$F_\nabla(X,Y)u = \nabla_X \nabla_Y s - \nabla_X \nabla_Y s - \nabla_{[X,Y]} s$$

for $X, Y \in \Gamma(TM)$, $s \in \Gamma(E)$.
The Leibniz rule for connections ensures that $F_\nabla(X,Y)(fs) = fF_\nabla(X,Y)s$, (and of course F_∇ is linear over $C^\infty(M)$ in the other two slots because $\nabla_{fX} s = f\nabla_X s$). Therefore, $F_\nabla \in \Omega^2(End(E))$ (i.e. $F_\nabla(X,Y) \in End(E)$).
We can formulate the curvature terms of the gluing cocycles and the local data of the connection $\{A_\alpha\}$. First consider the trivial vector bundle $E = M \times V \to M$. Then $\nabla = d + A$ where $A \in \Omega^1(End(E))$, i.e. a square matrix of 1-forms. We can calculate the curvature:

$$\begin{aligned}
F_\nabla(X,Y)s &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s \\
&= \nabla_X (ds(Y) + A(Y)s) - \nabla_Y (ds(X) + A(X)s) - (ds([X,Y]) + A([X,Y])s) \\
&= d(ds(Y))(X) + d(A(Y)s)(X) + A(X)ds(Y) + A(X)A(Y)s \\
&\quad -d(ds(X))(Y) - d(A(X)s)(Y) - A(Y)ds(X) - A(Y)A(X)s \\
&\quad -ds([X,Y]) - A([X,Y])s \\
&= d(A(Y))(X)s + A(Y)ds(X) + A(X)ds(Y) - d(A(X))(Y)s - A(X)ds(Y) - A(Y)ds(X) \\
&\quad -A([X,Y])s + A \wedge A(X,Y)s \\
&= d(A(Y))(X)s - d(A(X))(Y)s - A([X,Y])s + A \wedge A(X,Y)s \\
&= dA(X,Y)s + A \wedge A(X,Y)s
\end{aligned}$$

We conclude that $F_\nabla = dA + A \wedge A$ where dA indicates the nxn matrix of 2-forms which are the exterior derivatives of the entries of A, and $A \wedge A(X,Y) = A(X)A(Y) - A(Y)A(X)$ is an endomorphism of E.

In the case of a general bundle, ∇ restricts to $d + A_\alpha$ on the charts U_α, so on each chart $F_\nabla = dA_\alpha + A_\alpha \wedge A_\alpha$. We have the relation (*) on the overlaps between the charts, which can be used to show $F_\nabla = dA_\alpha + A_\alpha \wedge A_\alpha$ is the same on overlaps regardless of which chart we are working in. Therefore, by piecing together the local data, we obtain a global endomorphism valued 2-form F_∇ from the local data of the bundle and connection.

When the vector bundle E has a G-structure, namely it is the associated bundle to a principal G-bundle, the curvature is a 2-form valued in $Ad(E)$. [The adjoint action: $Ad : G \to Aut(\mathfrak{g})$, is defined by sending $g \in G$ to $Ad_g : \mathfrak{g} \to \mathfrak{g}$.]